https://www.linkedin.com/feed/update/urn:li:activity:7272496513446998016?utm_source=share&utm_medium=member_desktopSolve the system of equations:

$$\begin{cases} (q+r)(x+1/x) = (r+p)(y+1/y) = (p+q)(z+1/z) \\ xy+yz+zx = 1 \end{cases}$$

where p,q,r are positive real numbers.

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Solution.

Let $a := q+r, b := r+p, c := p+q$. Since $p,q,r > 0$ then a,b,c satisfy to triangle inequalities and, therefore, numbers a,b,c determiner a triangle ABC with sidelengths $a = BC, b = CA, c = AB$. Note that x,y,z have the same sign and since $xy+yz+zx$ and $(q+r)(x+1/x) = (r+p)(y+1/y) = (p+q)(z+1/z)$ are invariant with respect to transformation $(x,y,z) \mapsto (-x,-y,-z)$ we further assume that $x,y,z > 0$.

Let $\alpha := 2\tan^{-1}x, \beta := 2\tan^{-1}y, \gamma := 2\tan^{-1}z$. Since $x,y,z > 0$ then $\alpha,\beta,\gamma \in (0,\pi)$, $x+1/x = \dfrac{2}{\sin\alpha}$, $y+1/y = \dfrac{2}{\sin\beta}$, $z+1/z = \dfrac{2}{\sin\gamma}$, $xy+yz+zx = 1 \Leftrightarrow$

(1) $\quad \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} + \tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2} + \tan\dfrac{\gamma}{2}\tan\dfrac{\alpha}{2} = 1$ and

$(q+r)(x+1/x) = (r+p)(y+1/y) = (p+q)(z+1/z)$ can be rewritten in the form

(2) $\quad \dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$.

Now we will pay attention to the correlation (**1**)

We have (**1**)$\Leftrightarrow \tan\dfrac{\alpha}{2}\left(\tan\dfrac{\beta}{2} + \tan\dfrac{\gamma}{2}\right) = 1 - \tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}$.

Note that $\tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2} \neq 1$ because otherwise since $\beta,\gamma \in (0,\pi)$ we obtain $\tan\dfrac{\alpha}{2} = 0 \Leftrightarrow \alpha = 0$ (contradiction with $\alpha > 0$).

Thus, (**1**)$\Leftrightarrow \tan\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2}\right) = \dfrac{\tan\dfrac{\beta}{2} + \tan\dfrac{\gamma}{2}}{1 - \tan\dfrac{\beta}{2}\tan\dfrac{\gamma}{2}} \Leftrightarrow \tan\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2}\right) = \tan\left(\dfrac{\beta}{2} + \dfrac{\gamma}{2}\right) \Leftrightarrow$

$\dfrac{\pi}{2} - \dfrac{\alpha}{2} = \dfrac{\beta}{2} + \dfrac{\gamma}{2} \Leftrightarrow \alpha+\beta+\gamma = \pi$ (because $\dfrac{\pi}{2} - \dfrac{\alpha}{2}, \dfrac{\beta}{2} + \dfrac{\gamma}{2} \in \left(0, \dfrac{\pi}{2}\right)$).

Since $\alpha,\beta,\gamma \in (0,\pi)$ and $\alpha+\beta+\gamma = \pi$ then α,β,γ can be considered as angles of some triangle with correspondent sidelengths $\sin\alpha, \sin\beta, \sin\gamma$ which due to (**2**) is similar to triangle ABC.

Hence, $\alpha = A, \beta = B, \gamma = C$ and, therefore, $(x,y,z) = \left(\tan\dfrac{A}{2}, \tan\dfrac{B}{2}, \tan\dfrac{C}{2}\right)$ and $(x,y,z) = \left(-\tan\dfrac{A}{2}, -\tan\dfrac{B}{2}, -\tan\dfrac{C}{2}\right)$ all solutions of the system

$$\begin{cases} a(x+1/x) = b(y+1/y) = c(z+1/z) \\ xy+yz+zx = 1 \end{cases}.$$

It remains only to express $\tan\dfrac{A}{2}, \tan\dfrac{B}{2}, \tan\dfrac{C}{2}$ via p,q,r.

Let $s := \dfrac{a+b+c}{2} = p+q+r$ then $\tan\dfrac{A}{2} = \dfrac{r}{s-a} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\dfrac{qr}{p(p+q+r)}}$

and, cyclic, $\tan\frac{B}{2} = \sqrt{\frac{rp}{q(p+q+r)}}$, $\tan C = \sqrt{\frac{pq}{r(p+q+r)}}$.

So, $(x,y,z) = \pm\frac{1}{\sqrt{p+q+r}}\left(\sqrt{\frac{qr}{p}}, \sqrt{\frac{rp}{q}}, \sqrt{\frac{pq}{r}}\right)$ all solutions of original system.